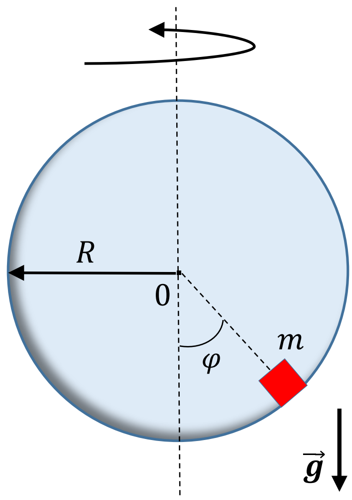
PHYS 101 Midterm Exam 2 Solution 2021-22-1

1. A block of mass m is placed on the inner surface of a hollow sphere of radius R as shown in the figure. The coefficient of static friction between the block and the inner surface of the sphere is μ_s. Initially the sphere and the block are both at rest.



(a) (5 Pts.) Draw a free-body diagram for the block.

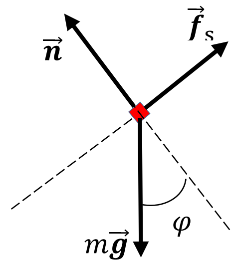
(a) (5 Pts.) What is the minimum value of the coefficient of static friction which keeps the mass from sliding down inside the sphere?

(b) (10 Pts.) If the sphere starts to rotate about the vertical axis passing through its center 0 and the rotation rate increases slowly, what will the period T of the sphere be when the block begins to slide?

(c) (5 Pts.) What is the condition on the coefficient static friction μ_s for the block to never slide no matter how fast the sphere is rotating?

Solution:

(a)



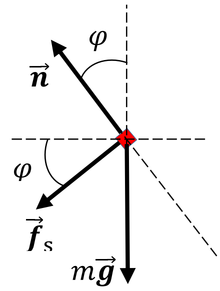
(b) Writing Newton's second law for the free body didagram of part (a), we have

$$mg\sin\varphi - f_s = 0\,, \qquad n - mg\cos\varphi = 0\,, \qquad f_s \le \mu_s n \quad \rightarrow \quad \mu_s \ge \tan\varphi.$$

(c) Free body diagram for the block when the sphere starts to rotate is given. Writing Newton's second law, we have

$$n\sin\varphi + f_s\cos\varphi = m\,a_{\text{rad}}\,, \;\; n\cos\varphi - f_s\sin\varphi - mg = 0\,.$$

$$a_{\text{rad}} = \frac{v^2}{R\sin\varphi}$$



Solving for the normal force and the friction force, we get

$$n = ma_{\text{rad}}\sin\varphi + mg\cos\varphi\ , \qquad f_s = ma_{\text{rad}}\cos\varphi - mg\sin\varphi\,.$$

Using

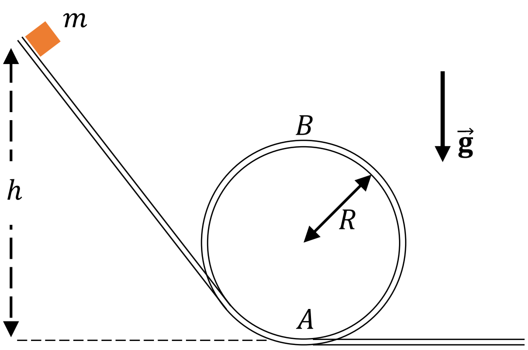
$$f_s \le \mu_s n \;\; \text{and} \;\; a_{\text{rad}} = \frac{4\pi^2 R\sin\varphi}{T^2}$$

we obtain

$$T \le 2\pi\sqrt{\frac{R\sin\varphi}{g}\left(\frac{\cos\varphi - \mu_s\sin\varphi}{\sin\varphi + \mu_s\cos\varphi}\right)}\,.$$

(d) If $\cos\varphi - \mu_s\sin\varphi < 0$ then T is not real, meaning that if $\mu_s \ge \cot\varphi$ the block does not slide no matter how fast the sphere is rotating.

2. A small block of mass m is released from rest at a height h above the horizontal, and slides without friction along the looped track shown in the figure. Part of the track is a circular loop of radius R.



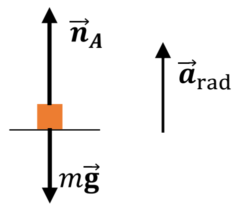
(a) (5 Pts.) Calculate the normal force on the block exerted by the track at the bottom of the loop (point A).

(b) (5 Pts.) Calculate the normal force on the block exerted by the track at the top of the loop (point B).

(c) (15 Pts.) Determine the minimum release height h if the block is to remain on the track at all times, even at the very top of the loop.

Solution:

(a) Total mechanical energy is conserved. $E_{\mathrm{i}} = mgh\,,\ E_A = \frac{1}{2}mv_A^2 \quad \rightarrow \quad v_A^2 = 2gh.$

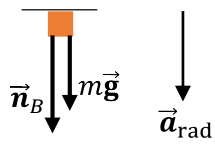


Drawing the free body diagram of the block at point A, we find

$$n_A - mg = m\vec{\boldsymbol{a}}_{\mathrm{rad}}\,, \qquad \vec{\boldsymbol{a}}_{\mathrm{rad}} = m\frac{v_A^2}{R} \quad \rightarrow \quad n_A = mg\left(1 + \frac{2h}{R}\right).$$

(b) Conservation of total mechanical energy at point B means $E_B = \frac{1}{2}mv_B^2 + mg(2R) \quad \rightarrow \quad v_B^2 = 2g(h - 2R).$

Drawing the free body diagram of the block at point B, we find



$$n_B + mg = m\vec{\boldsymbol{a}}_{\mathrm{rad}}\,, \qquad \vec{\boldsymbol{a}}_{\mathrm{rad}} = m\frac{v_B^2}{R} \quad \rightarrow \quad n_B = mg\left(\frac{2h}{R} - 5\right).$$

(c) To complete the loop without loosing contact with the track, the normal force on the block at point A must be equal to or grater than zero. So

$$n_B \geq 0 \quad \rightarrow \quad mg\left(\frac{2h}{R} - 5\right) \geq 0 \quad \rightarrow \quad h \geq \frac{5}{2}R\,.$$

3. A bead of mass m can slide without friction on a horizontal wire. The bead is connected to a fixed nail by an ideal spring of spring constant k and relaxed length L. The nail is $L/2$ away from the wire as shown in the figure. We stretch the spring by pulling the bead to the right to a distance X_m from the origin and then let it go.

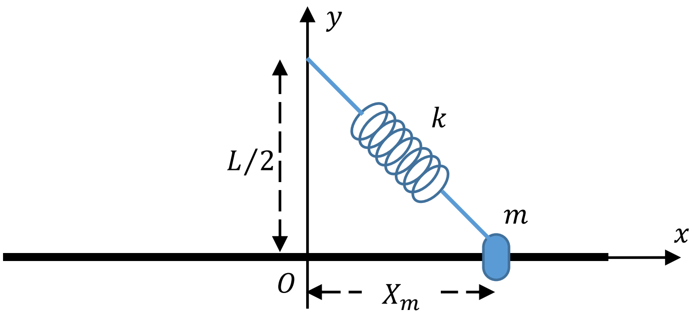
(a) (5 Pts.) What is the minimum X_m so that the mass can pass to the left side (i.e. $x < 0$)?

(b) (5 Pts.) What will be the maximum kinetic energy of the bead during its motion in terms of k, L and X_m?

(c) (5 Pts.) What is the spring potential energy as a function of x?

(d) (7 Pts.) Give the coordinates of the three equilibrium points of the bead.

(e) (3 Pts.) For each equilibrium point indicate if it is stable or unstable equilibrium.



Solution:

(a) During the motion of the bead its total mechanical energy is conserved.

$$E(x=0) = \frac{1}{2}k\left(\frac{L}{2}\right)^2 , \qquad E(x=X_m) = \frac{1}{2}k\left(\sqrt{X_m^2 + \frac{L^2}{4}} - L\right)^2 .$$

To pass the point $x = 0$ to the right we need to have $E(x = X_m) \geq E(x = 0)$.

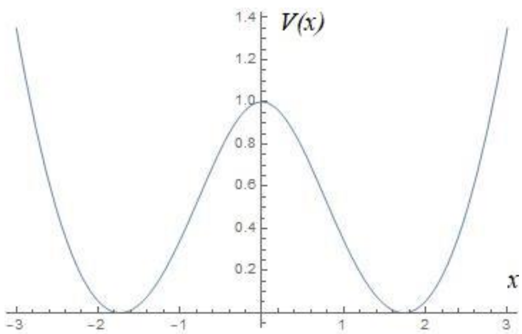
$$\frac{1}{2}k\left(\sqrt{X_m^2 + \frac{L^2}{4}} - L\right)^2 \geq \frac{1}{2}k\left(\frac{L}{2}\right)^2 \quad \rightarrow \quad X_m \geq \sqrt{2}L .$$

(b) As the bead moves, the length of the spring will be equal to its natural length at some point. Since at that point the potential energy in the spring is zero, all of total mechanical energy will be in kinetic form, meaning maximum kinetic energy. Hence

$$K_{\max} = E(X_m) = \frac{1}{2}k\left(\sqrt{X_m^2 + \frac{L^2}{4}} - L\right)^2 .$$

(c)

$$V(x) = \frac{1}{2}k\left(\sqrt{x^2 + \frac{L^2}{4}} - L\right)^2 .$$

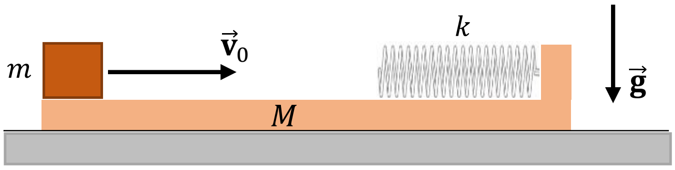


(d) Equilibrium point means

$$\frac{dV}{dx} = 0 \quad \rightarrow \quad k\left(\sqrt{x^2 + \frac{L^2}{4}} - L\right)\left(\frac{x}{\sqrt{x^2 + \frac{L^2}{4}}}\right) = 0 \quad \rightarrow \quad x = 0 , \qquad x = \pm\frac{\sqrt{3}}{2}L .$$

(e) $x = 0$ is an unstable equilibrium point, while $x = \pm\frac{\sqrt{3}}{2}L$ are stable equilibrium points.

4. An L-shaped block of mass M is free to slide on a horizontal surface. It has a massless horizontal spring with stiffness constant k attached to one end. A small block of mass m is placed on top of the L-shaped block as shown in the figure. Initially both blocks are at rest, and all surfaces are frictionless.



(a) (8 Pts.) The small block is given an initial velocity $\vec{\mathbf{v}}_0$ and, after sliding on the L-shaped block, collides with the spring. What will be the maximum compression of the spring?

(b) (10 Pts.) What will be the speed of the small block relative to the L-shaped block when it loses contact with the spring as the spring pushes it back?

(c) (7 Pts.) What will be the speed of the center of mass of the system after the spring pushes the small block back?

Solution: (a) Maximum compression of the spring occurs when both objects have the same speed.

$$p_i = mv_0\,, \qquad p_f = (m+M)V \quad \rightarrow \quad V = \frac{mv_0}{m+M}$$

$$E_i = \frac{1}{2}m{v_0}^2\,, \qquad E_f = \frac{1}{2}(m+M)V^2 + \frac{1}{2}kx_{\max}^2 \quad \rightarrow \quad x_{\max} = v_0\sqrt{\frac{mM}{k(m+M)}}\,.$$

(b) During the collision momentum relative to the horizontal surface is conserved, and energy is conserved. Letting v_1 and v_2 denote final velocities with respect to the horizontal surface of the small block and of the L-shaped block respectively, we have

$$mv_0 = mv_1 + Mv_2\,, \qquad \frac{1}{2}mv_0^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2\,, \qquad \text{or, equivalently } v_0 = v_2 - v_1\,.$$

Solving for v_1 and v_2, we get

$$v_1 = \frac{m-M}{m+M}v_0\,, \qquad v_2 = \frac{2m}{m+M}v_0\,.$$

Velocity $v_{1/2}$ of the small block relative to the L-shaped block when it loses contact with the spring will be

$$v_{1/2} = v_{1/E} + v_{E/2} = v_1 - v_2 \quad \rightarrow \quad v_{1/2} = -v_0\,.$$

So, the speed is $|v_{1/2}| = v_0$.

(c) Since the velocity of the center of mass of the system does not change during a collison, we have

$$v_{\mathrm{CM}} = \frac{m\,v_0}{m+M}\,.$$